JOURNAL OF MATERIALS SCIENCE 24 (1989) 1559–1569

Statistical inference about stress concentrations in fibre–matrix composites

LINDA C. WOLSTENHOLME, RICHARD L. SMITH
Department of Mathematics, University of Surrey, Guildford GU2 5XH, Surrey, UK

In previous papers about statistical properties of composite materials, assumptions have been made about both the statistical and mechanical properties of the component fibres and matrix, and these have been used to calculate properties of the composite. The results are sensitive to the assumed stress concentration factors and length of stress overload region, and there remains considerable interest in characterizing them. In this paper, it is shown how experimental data on the fibres and composites may be used to make inferences about these properties of the material. The statistical technique employed is numerical maximum likelihood, but this involves detailed combinatorial calculations and is therefore highly computationally intensive. The method is illustrated using experimental data on hybrid composites consisting of carbon fibre tows embedded in glass–epoxy composite, particular emphasis being placed on the consequences of varying the distance between the carbon fibre tows.

1. Introduction

Theories regarding the behaviour of a collection of fibres of equal length held in parallel originated with Daniels in 1945 [1]. His model assumed that, if some fibres fail, the load is redistributed equally over all surviving fibres. In recent years, models involving local load sharing and the chain of bundles model have been extensively studied, some relevant references being Harlow and Phoenix [2–5], Manders and Bader [6, 7], Batdorf [8], Bader and Priest [9] and Smith *et al.* [10]. In all these papers assumptions of two types have been involved. First, individual fibres are assumed to have random strength, the strength distribution most frequently being taken to be of Weibull form. Secondly, assumptions are made about the stress concentrations that surround individual fibre failures. These assumptions are particularly critical in the case of hybrid composites, as recent analyses by Bader and co-workers [11, 12] have shown. Whereas the Weibull distribution for individual fibre strength may be inferred from tests on single fibres, there is no direct method to measure the stress concentration factors.

The motivation behind the present paper is that if detailed data are available concerning the positions and failure stresses of individual breaks in the composite, then much can be learned about the stress concentrations. This is based on a new method of statistical analysis which, in contrast to earlier statistical methods which were mainly concerned with estimating the Weibull distribution, attempts to model the whole process of the appearance of flaws in the material. The statistical ideas are closely connected with methods for inference from point processes, e.g. [13], though the models involved are so specialized that very little use has been made of standard methodology.

The experimental data analysed in the paper are taken from experiments on glass–carbon hybrids, reported previously by Bader and Pitkethly [11]. The material is a glass-epoxy composite into which tows of carbon fibre have been inserted. Typically there are seven carbon tows in each specimen, and different distances between tows have been used: 1.5, 1.0, 0.5 mm and touching. These specimens have been tested in a Instron machine and the positions and failure strains of individual breaks in the carbon tows have been noted. The result is a failure pattern of the form illustrated in Figs 1 to 3. We can see at once that, when the carbon tows are only 0.5 mm apart, failure tends to occur across the material, whereas in the 1.5 mm case the pattern of failures in the different tows shows no obvious correlation. The touching case (not shown) exhibits straight breaks across the material. It may be inferred that, in the touching case, there is a high degree of stress overload, whereas in the 1.5 mm case the degree of stress overload between carbon tows is small if there is any at all. The question is how to quantify such observations.

We propose in this paper a new method of statistical analysis by which it is possible to make inferences about the magnitudes of the stress concentration factors. The method is based on the principle of maximum likelihood, and requires detailed computations of individual sequences of fibre failures. For this reason it is highly computationally intensive. The method yields numerical estimates of several parameters describing the pattern of stress concentrations, but there is considerable uncertainty about these parameter estimates as reflected in the confidence intervals derived. Nevertheless, the method allows us to evaluate the differences among stress concentration patterns at different fibre spacings, and to make some inferences about the lengthwise distribution of stress concentrations in the neighbourhood of a failed fibre.

0022–2461/89 $03.00 + .12

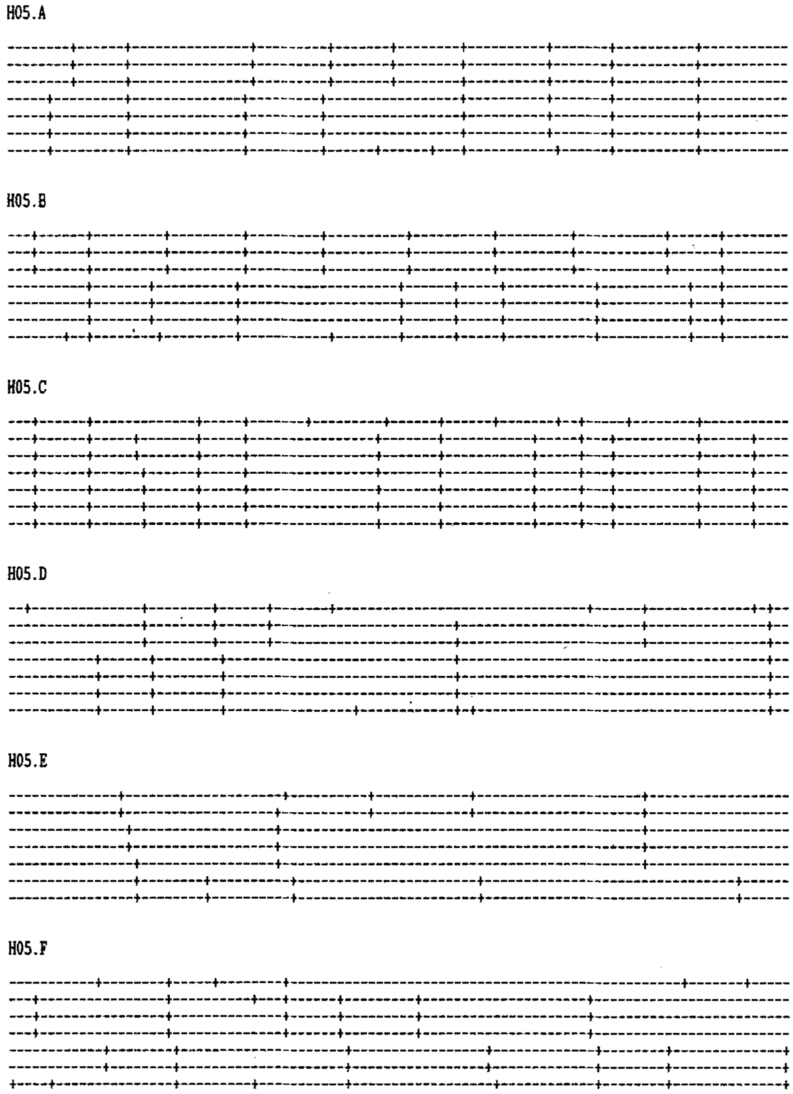

Figure 1 Patterns of breaks in tows with 0.5 mm spacing.

2. Statistical models for single-fibre strength

The long established Weibull model for single-fibre strength, introduced by Weibull [14], rests on the assumption that failure is due to flaws which occur independently and randomly along the length of the fibre. The two-parameter Weibull distribution gives the survivor function, i.e. the probability that a fibre survives stress x, as

$$\bar{F}_a(x) = \exp[-a(x/x_1)^w] = \exp[-(x/x_a)^w]$$

where a is the length of the fibre, x_1 the characteristic stress of the fibre at length l, $x_a = x_1 a^{(-1/w)}$ the characteristic stress at length a, and w is the Weibull shape parameter. This distribution is commonly deduced from the "weakest-link" concept, i.e. the assertion that a fibre is only as strong as its weakest portion. Long fibres must be weaker than short fibres because in a long fibre the probability of encountering a flaw is greater than in a short fibre.

Suppose a fibre of length a consists of n segments of length d. Then the probability that the fibre survives stress x must equal the probability that all the segments survive stress x. If all segments can be regarded as independent then

$$\bar{F}_a(x) = (\bar{F}_d(x))^n.$$

The wide popularity of the Weibull distribution arises from the fact that it is consistent with this relation, and is a simple two-parameter function which is found to be consistent with strength data for a wide variety of materials. Although other forms of distribution have been considered, including the three-parameter Weibull and various bimodal forms, only the standard Weibull distribution will be considered in this paper.

Suppose now that a fibre of length l is known to fail at some point between stress x and the higher stress y. The probability of this event is

$$\text{prob(survives } x) - \text{prob(survives } y)$$
$$= \bar{F}_l(x) - \bar{F}_l(y)$$
$$= F_l(y) - F_l(x) \quad \text{where } F = 1 - \bar{F}.$$

In the limiting case $y \to x$, this reduces to $f_l(x)\,dx$ where $f_l(x) = dF_l(x)/dx$ is the probability density function and $dx = y - x$.

Now suppose a fibre fails whilst under the influence of transferred load due to the failure of adjacent fibres. This enhancement of effective load may be reflected in the application of a stress concentration factor k. When the applied stress is x the fibre experiences stress kx. As failures around a fibre progress, this stress concentration factor will increase from its initial value of 1.

Suppose it is observed that a fibre, currently surviving stress kx, fails when the stress concentration factor

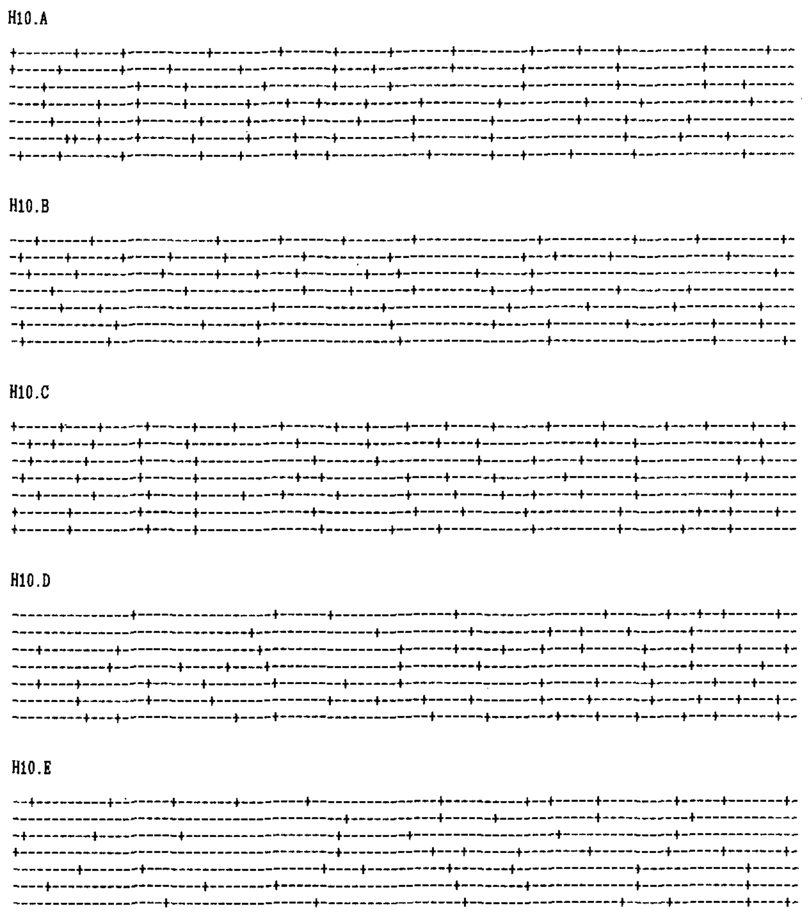

Figure 2 Patterns of breaks in tows with 1.0 mm spacing.

is increased from k to $k^{\sim}$. We do not know the exact strength of the fibre, but only that it lies between kx and $k^{\sim}x$. The probability of this event is

$$F_l(k^{\sim}x) - F_l(kx)$$

Suppose, in contrast, that the same fibre survives under the increase of stress concentration factor from k to $k^{\sim}$ but, as the stress on the system is gradually increased from x, it fails at a stress (on system) y. The probability of this outcome may be represented as

$$k^{\sim}f_l(k^{\sim}y)\ \mathrm{d}y$$

where the factor $k^{\sim}$ in front reflects the fact that in increment of dy in system stress results in an increment $k^{\sim}$ dy in fibre stress.

Such calculations underlie the principle of the method: we are able to calculate probabilities of observed patterns of failed and unfailed fibres, as a function of unknown stress concentration parameters. The method of maximum likelihood then permits us to make inferences about those parameters.

This section is concluded with a brief outline of the method of maximum likelihood, which is by far the most widely used method for fitting complex statistical models.

Suppose observed data $x_1, x_2, \ldots, x_n$ can be thought of as realizations of a set of random variables $X_1, X_2, \ldots, X_n$, whose probability distributions are known except for a finite number of unknown parameters. There may be "censored" data, for example, in the case of a set of fibres, if fibre i did not fail within the duration of the experiment then x_i may represent the maximum stress which the fibre is known to have survived. The likelihood function is defined to be the joint probability of the observed data, being expressed as a function of the unknown parameters. The maximum likelihood estimators are then those values of the parameters which maximize the likelihood function. Note that the whole procedure depends on the assumed parametric model being correct, and it is usual to combine the method with checks on the correctness of that model.

The likelihood function may be represented as $L(\boldsymbol{\theta}; \boldsymbol{x})$, where $\boldsymbol{\theta}$ is a vector of unknown parameters and $\boldsymbol{x}$ is the data. In most complex models, the estimate $\hat{\boldsymbol{\theta}}$ which maximizes $L(\boldsymbol{\theta}; \boldsymbol{x})$ must be obtained numerically, therefore requiring a suitable algorithm for numerical optimization. The matrix of second-order derivatives of $-\log L$, evaluated at the maximum likelihood estimate $\hat{\boldsymbol{\theta}}$, is termed the observed information matrix. Its inverse is an approximation to the variance–covariance matrix of the parameter estimates. All these concepts are treated in detail in standard texts on statistical methods. Watson and Smith [15] have several specific examples of maximum likelihood being applied to fit extensions of the standard Weibull model.

For the analysis proposed here, the observed data consist of the positions of all breaks of individual carbon tows within the glass–epoxy matrix, together with the stresses at which the individual breaks occurred. The unknown parameters may consist of the Weibull scale and shape parameters of the individual tows, and the parameter F governing the stress concentration factors. The likelihood function then involves calculating the probability of the observed outcome as a function of these unknown parameters. These calculations are sufficiently complex that

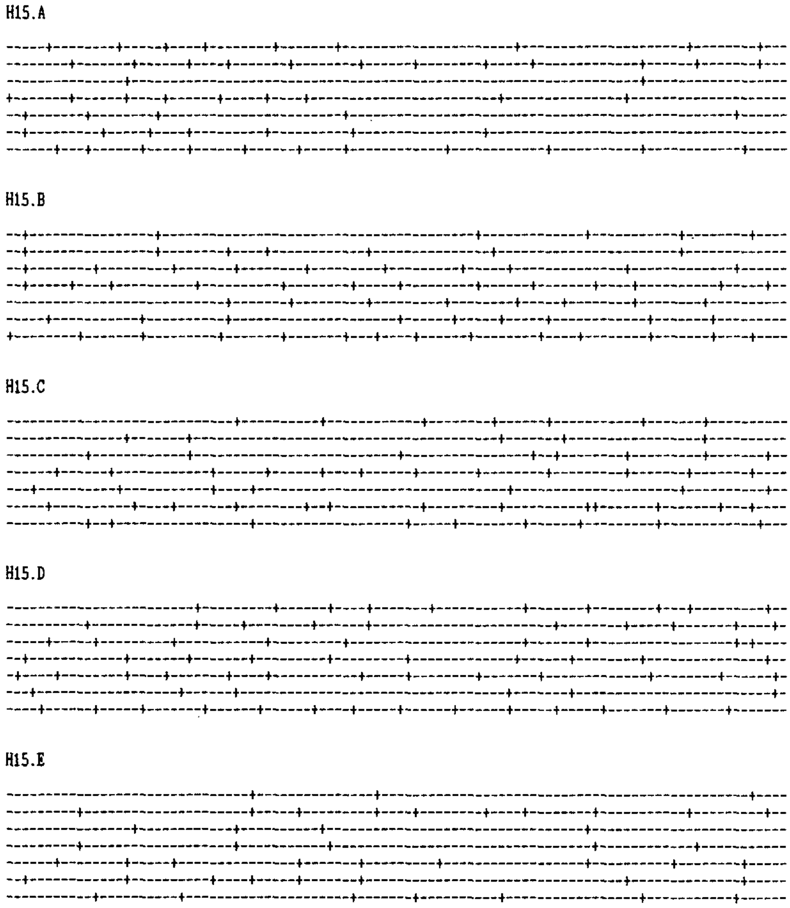

Figure 3 Patterns of breaks in tows with 1.5 mm spacing.

considerable computer time is involved merely in evaluating the likelihood function, which then has to be maximized with respect to the unknown parameters. Consequently, the maximization itself is a somewhat *ad hoc* process.

3. Models for fibrous composites

Many of the theoretical models for fibrous composites, e.g. those studied by Harlow and Phoenix [2–5], are concerned with fibres arranged in a linear equally spaced array. The load on any unfailed fibre depends on how many failed fibres are adjacent to it. If this number is r then the load concentration factor k is given by

$$k = 1 + g(r)$$

The function $g(r)$ may take many forms. In the papers of Harlow and Phoenix, it was assumed that $g(r) = r/2$, corresponding to the assumption that all load on a group of consecutive failed fibres is transferred to the two nearest neighbours. It is recognized, however, that this represents an extreme situation. The analysis to follow is based on the glass–carbon hybrid experiments described above, in which the "fibres" consisted of carbon–fibre tows embedded within a glass–epoxy matrix material. Following Bader and Pitkethly [11], it is assumed that

$$g(r) = (r)^{1/2}/F$$

where F is a parameter, termed the load-sharing factor, which reflects the degree to which the load of a failed fibre has been transferred to other fibres. The parameter F is expected to increase with the inter-fibre distance, being effectively infinite if this distance exceeds four to five tow diameters. This reflects the notion that, at such a wide separation between tows, all the excess load is absorbed by the glass–epoxy matrix and there is no load transfer between tows.

It is accepted that in practice all sharing fibres do not bear an equal load increment, and that the actual stress concentrations are not as localized as the model implies. However, the simplified model is considered to represent the most essential features of the situation and makes the following calculations possible.

In accordance with the "chain of bundles models" [2–5], a set of N parallel fibres (a bundle) is considered as a set of m independent sub-bundles length d. So $md = l$ and d is often referred to as the ineffective length. Fig. 4 illustrates the model.

With data for the failures recorded in such a bundle, a contribution to the likelihood function, L_i say, may be calculated for the ith sub-bundle. The overall likelihood function is obtained by multiplying together the contributions from the sub-bundles.

As an example, suppose a sub-bundle provides the following data:

Fibre	1	2	3	4	5	6	7
Status	0	1	0	0	1	1	0
Failure stress	x_1		x_2	x_2			x_3

Here the fibre "status" is 0 if failed, 1 if unfailed. The failure stresses represent the stress on the bundle when failure occurred, and it is assumed that the maximum stress at which the remaining fibres survived is known, say x_M.

Under the given load-sharing rule, appropriate

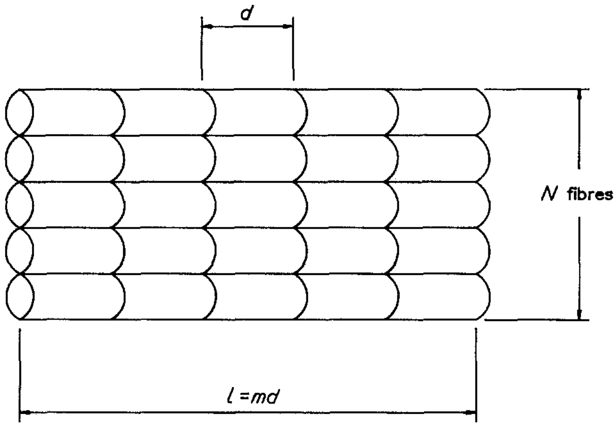

Figure 4 Chain-of-bundles model.

load-scaling factors may be calculated for each fibre, whether it be for the point of failure, an interval in which failure occurred, or when the fibre survived the final stress. The joint probability of this set of outcomes is proportional to

$$k_1 f(k_1 x_1)\bar{F}(k_2 x_M)\,[\ldots]\bar{F}(k_5 x_M)\bar{F}(k_6 x_M) k_7 f(k_7 x_3)$$

where we write f and F in place of f_d and F_d, respectively, and $\bar{F} = 1 - F$. Here $k_1, \ldots, k_7$ represent the stress concentration factors on fibres $1, \ldots, 7$ in the configuration illustrated. The term denoted $[\ldots]$ refers to the probability density for the event of two adjacent fibres apparently failing simultaneously at the same stress. This phenomenon is discussed next.

Suppose a group of n fibres is observed to have failed together at stress x. It is assumed that failure was initiated by just one of the group, and that the others followed as a result of the increased load placed upon them. For $n = 2$ there are only two possibilities: either fibre 1 was the initiator, or fibre 2 was. So the probability density for this event is

$$\text{prob}(1 \text{ then } 2) + \text{prob}(2 \text{ then } 1).$$

The order of failure will affect the load concentration factors, at failure or over an interval during which failure occurred. The probability density above will take the form

$$k_1 f(k_1 x)\,[F(k_2{}^{\sim} x) - F(k_2 x)] + k_2 f(k_2 x) \times [F(k_1{}^{\sim} x) - F(k_x)].$$

Here k_1 and k_2 are the initial stress concentration factors on fibres 1 and 2, $k_1{}^{\sim}$ is the stress concentration factor on fibre 1 after fibre 2 has failed, and $k_2{}^{\sim}$ is the stress concentration factor on fibre 2 after fibre 1 has failed. For $n = 3$ the possible sequences of failure are more numerous, and further introduces the situation where a failure may directly cause more than one other failure.

e.g. X X X 1 2 3 X

where 1, 2, 3 denote the subset of fibres which fail at a particular stress and the other fibres (denoted X) may be either failed or not. One possible sequence of failure is that 2 fails under its initial load, 3 fails under the overload from 2, and finally 1 fails as a result of the overload from 2 and 3. Another possibility is that the fibres fail in the order 2, 1, 3; yet another is that 2 fails first and then both 3 and 1 fail under the overload from 2. These and all other possibilities must be computed separately, the resulting probability density being a sum over disjoint sequences of failure. The probability density corresponding to each possible sequence will take the form

$$k_2 f(k_2 x)\,[F(k_3{}^{\sim} x) - F(k_3 x)]\,[F(k_1{}^{\sim} x) - F(k_1 x)]$$

but the values of $k_3{}^{\sim}$, k_3, $k_1{}^{\sim}$, k_1 will vary according to the pattern of failure.

In general, if we let M_i = the number of different patterns of failure, starting with fibre i, k_i = load concentration factor at failure for fibre i, $k_j(m)$ = load concentration factor on fibre j when it is known to have survived in failure pattern m, $k_j{}^{\sim}(m)$ = load concentration factor on fibre j when it is known to have failed in failure pattern m, then the probability density for the simultaneous failure of n fibres at stress x is

$$\prod_{i=1}^{n} k_i f(k_i x) \left\{ \sum_{m=1}^{M_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} [F(k_j{}^{\sim}(m)x) - F(k_j(m)x)] \right\}$$

For each set of values of the unknown parameters, the program calculates the sum of such probability densities over all sequences of failure consistent with the data. The resulting sum is L_i, the contribution to the likelihood for sub-bundle i.

The choice of the ineffective length d is arbitrary. It is intended to represent the distance in the direction of the composite over which the stress concentrations occur. There is considerable uncertainity as to the true value, and also as to how much it might vary with stress, but it is assumed here to be constant and a number of different values have been tried in the region of 5 to 10 fibre diameters as suggested by the early work of Rosen [16] and Zweben [17].

A guide to the choice of d is that, on dividing a bundle into sub-bundles of length d, it is desirable that each should have only one break per fibre, and breaks which might be related should be in the same sub-bundle, to support the independence notion. Clearly this latter condition cannot be guaranteed, and it must be borne in mind that the chain of bundles model is only an approximation.

The likelihood function for the complete bundle is a product across sub-bundles, i.e.

$$L_1 L_2 L_3 L_4, \ldots, L_m$$

but it is more convenient to use log likelihood

$$\sum_{i=1}^{m} \log L_i = L \text{ say}$$

Here L is primarily being considered as a function of F and d, but the Weibull parameters x_1 and w are also of interest. The strategy that has been adopted is to fix d and one or two of the other parameters and estimate the rest.

4. Experimental details

The experimental data which we have analysed is taken from the unpublished work of Pitkethly [18]. The composite was formed from carbon fibre, glass fibre and epoxy resin. The carbon fibre was taken from a spool of 1000-filament tow of Celion 1000. The fibre diameter is approximately 8 μm, and 1000 such fibres closely packed together to form a tow have a diameter of approximately 0.3 mm. The ensuing analysis treats the tow as though it were a single fibre of this diameter.

The glass fibre was an E-glass supplied in rovings of approximately 1750 and 3500 filaments, the fibres approximately 20 μm diameter. The resin was formulated from a standard bisphenonol-A epoxy cured with nadic methyl anhydride with an amine accelerator.

The hybrid test pieces were made by taking lengths of partially cured resin-impregnated carbon tows and incorporating them at different separations into sheets formed from unidirectional glass-fibre rovings. These were then impregnated with resin and cured.

Test coupons were cut from the sheets, 100 mm long, 2 mm thick, and wide enough to contain seven carbon tows. Three different spacings were used (measured tow centre to two centre): 0.5, 1.0, and 1.5 mm. A sheet was prepared for each spacing and each produced five or six test pieces labelled randomly, A, B, C, . . . , etc.

The strain to failure of the carbon fibre is approximately 1% and that of the glass fibre, 3%, so that on loading the carbon tows break first, the glass–epoxy "matrix" remaining intact. The glass–epoxy constrains the tows so that at a certain distance (d, effectively) away from the break, the full stress is restored and further breaks can be induced. It was possible to record the position and strain corresponding to each fracture of the carbon ligaments, positions to the nearest millimetre up to strains between 1% and 2%. At higher strain there is a risk of the glass fibre failing.

The patterns of breaks for the 0.5, 1.0 and 1.5 mm spacings are shown in Figs 1 to 3. At 0.5 mm, most of the breaks occur straight across the bundle, implying that a single break has resulted in the whole subbundle breaking. At 1.0 mm, the pattern of breaks is much more random, though there are still some breaks which go all the way across the specimen. At 1.5 mm, the pattern of breaks seems completely random. We would expect these different patterns to be reflected in different values of the load-sharing parameter F. In the case of touching tows (separation 0), all the failures occurred straight across the whole specimen.

Table I is included to give an indication of the numbers of breaks in the specimens and the range of failure strains.

5. Results of the statistical analysis

The first set of data to be analysed was the 1.5 mm data. The three parameters to be estimated are w, x_l (or equivalently x_a for any given a) and F. Initially w and x_l are assumed known, based either on the results of previous experiments or on analyses of initial failures (i.e. the first failure in each of the 100 mm tows) in the current experiment. The estimation based on initial failures is, in effect, assuming that these are independent from fibre to fibre, an assumption that seems reasonable in the case of the 1.5 mm spacings, but not for the other spacings in which the dependence between the fibres is more obvious. Later we shall consider joint estimation of x_l and F on the assumption that neither is known initially. The units of measurement for length are millimetres.

Maximum likelihood estimates of the Weibull parameters based on initial failures are $\hat{w} = 31.04$, $\hat{x}_{100} = 3.846$ GPa and hence $\hat{x}_l = 4.461$ GPa. The value for $\hat{x}_{100}$ exceeds the estimate 3.77 GPa quoted by Bader and Pitkethly [8] based on single-fibre tests, but is consistent with the value quoted in Pitkethly's thesis [18] for the present experiment. The difference would appear to be due to different samples of material being used for the two experiments. Based on these Weibull estimates, we also estimate $\hat{x}_{2.5} = 4.35$ GPa, $\hat{x}_4 = 4.28$ GPa, and take these as Weibull characteristic values for analyses based on ineffective lengths $d = 1$, 2.5 and 4 mm.

Next, the parameter F was estimated assuming x_d and w known as above. Maximization of the likelihood function yields a maximum likelihood estimate $\hat{F}$. A 95% confidence interval for F may be formed of all values for which 2 log $L(F)$ is within 3.84 of its maximum. This is derived from the chi-squared

TABLE I Classification of data sets by inter-tow distance, ranges of breaking strains and numbers of breaks

Inter-tow distance (mm)	Data set	Maximum %	Minimum %	No. Breaks
1.5	A	1.899	1.633	55
	B	1.936	1.697	67
	C	1.880	1.667	61
	D	1.859	1.606	72
	E	1.885	1.703	46
1.0	A	1.804	1.506	85
	B	1.785	1.612	66
	C	1.782	1.547	91
	D	1.750	1.493	74
	E	1.752	1.605	56
0.5	A	1.826	1.641	61
	B	1.843	1.660	68
	C	1.933	1.593	84
	D	1.836	1.695	43
	E	1.805	1.644	29
	F	1.850	1.576	48

TABLE II

Data set	$\hat{F}$	Confidence interval
A	> 1000	
B	100	[49, ∞]
C	85	[29, ∞]
D	46	[23, ∞]
E	50	[25, ∞]

approximation to twice the log likelihood function, which is a standard method for forming approximate confidence intervals from the likelihood function.

For $d = 1$, estimates based on the five specimens depicted in Fig. 1 were as shown in Table II. The very wide confidence intervals (upper limit infinity) reflect the fact that the information obtained from a single data set is not sufficient to determine F at all precisely, and the inclusion of $F = \infty$ in the confidence interval means simply that a hypothesis of no stress overload is not rejected by the data.

We can obtain more meaningful results, however, by combining the samples into a single likelihood function. Such a procedure implicitly assumes that the parameters being estimated are the same across all the samples, but in this case such an assumption seems reasonable because all the specimens were obtained from the same batch of fibre. Combining the five specimens yields a point estimate $\hat{F} = 100$, confidence interval [51, 460].

Similar calculations for $d = 2.5$, 4 yield $\hat{F} = 120$, conf. int. [75, 270]; and $\hat{F} = 170$, conf. int. [97, 520], respectively. In each of these cases, finite confidence intervals for F are obtained, but they are still very wide. This is, however, to be expected, because it is evident that the amount of dependence between the fibres is slight and precise estimates of F are unlikely to be achieved. The problem of reconciling the estimates from different values of d raises separate issues, which will be discussed after the analysis of the data on 1.0 and 0.5 mm spacings.

We turn now to the 1.0 mm data. In this case, estimation of the Weibull parameters based on initial failure data is less reasonable, because the degree of dependence between the fibres is far higher. However, there were differences in the inherent strengths of the sheets from which the test coupons were made. In particular the 1.0 mm spaced material was of lower strength than the others. The strains at which breaks started to occur were clearly lower than for the other test pieces, and at the start of loading these would not be affected by load-sharing. Taking into consideration the apparent non-comparability of the 1.0 and 1.5 mm data, it was decided at first to proceed as in the case of the 1.5 mm data, i.e. by estimating the Weibull parameters from initial failures and then estimating F by our maximum likelihood procedure.

TABLE III

d	x_d
1.0	4.12
2.0	4.026
2.5	3.996
4.0	3.934
5.0	3.905

TABLE IV

Data set	$\hat{F}$	Confidence interval
A	130	[72, 383]
B	149	[75, 433]
C	24	[22, 27]
D	∞	[74, ∞]
E	83	[43, 262]
Combined	48	[42, 56]

1.0 mm spacing, $d = 2.0$ mm, $w = 30$, $x_d = 4.026$.

The 35 stresses gave parameter estimates $\hat{w} = 44.65$, $\hat{x}_{100} = 3.53$. The increase in w is accounted for by a degree of dependence which will tend to reduce the variance of fibre strengths. For the same reason a reduced estimate of x_{100} would be expected but not by such a large amount in comparison with the earlier value 3.85. In view of this it was felt that 3.53 was the more representative value, but w was kept at 30 for consistency with earlier results.

Using a characteristic stress marginally above 3.53 for 100 mm gauge length yields a characteristic stress of 4.12 for unit length fibres under weak-link scaling, taking w as 30. This, in turn, yields the following characteristic stresses for a variety of sub-bundle lengths as shown in Table III. For the moment, d is fixed at 2.0.

The Table IV shows estimates of F for each data set and also for all data combined together. This latter case assumes that the parameters are the same for each specimen – a more questionable assumption in this case because there is evidence that specimen C has smaller F than the others. This might be anticipated, because it is evident from Fig. 2 that set C has a greater degree of adjacency in its breaks than the other sets. For set D, the likelihood is increasing all the way to infinity and so we obtain $\hat{F} = \infty$ as the maximum likelihood estimate.

Similar calculations for $d = 1.0, 2.5, 4$ yield: $\hat{F} = 80$ conf. int. [57, 122]; $\hat{F} = 115$, conf. int. [85, 175]; $\hat{F} = 67$, conf. int. [58, 78]; respectively, where in this case only the combined results have been given.

So far, the estimation problem has been treated as one of estimating F under the assumption that both x_d and w are known. A difficulty with the 1.0 mm data, however, was the initial estimation of x_d, which either required using data from other experiments or making a Weibull analysis based on initial failures. The first approach is not adequate because of the very clear evidence of noncomparability between the specimens at different spacings, and the second approach is limited by the assumption that initial failures in the distinct fibres are independent.

An alternative approach is to estimate all three parameters (w, x_d and F) by a three-parameter maximization of the likelihood function based on all the breaks in the 1.00 mm specimens. This is limited by computational considerations, the difficulty of obtaining more than a few realizations of the likelihood function preventing a full-scale optimization.

The following compromise was adopted. Noting that the value of F seemed more dependent on x_d than w, the value $w = 30$ was fixed for consistency with results for other spacings, and the likelihood function

TABLE V

x_d	F								
	28	30	32	34	36	38	40	42	44
4.00							725.9		
4.02									
4.04						679.6	678.9		
4.06			676.3	673.2	671.4	670.5	670.3	670.7	671.4
4.08		672.5	669.7	668.4	668.1	668.5	669.4	670.7	672.2
4.10	674.5	671.9	671.0	671.2	672.2	673.8	675.6	677.7	
4.12		678.3	679.0						

1.00 mm spacing, $d = 2.0$, $w = 30$.

evaluated over a sparse grid of (x_d, F) values. The results given in Table V were obtained for the negative log likelihood. The minimum is 668.1 corresponding to the estimates $\hat{F} = 36$ and $\hat{x}_d = 4.08$ and hence $\hat{x}_l = 4.175$. Thus $\hat{F}$ is somewhat lower and $\hat{x}_l$ slightly higher than the estimates obtained earlier. An approximately 95% confidence region may be defined as consisting of all (F, x_d) combinations with a negative log likelihood within 1.9 of the minimum, i.e. less than 670.0 in Table V. This includes an appreciable range of F values (32 to 40) but allows much less variability in the x_d-direction.

The same procedure was repeated for different values of d with the following point estimates (Table VI). These results are not satisfactory. The values of F do not vary in a consistent way (we would expect them to increase with d – see Section 6 below) and the estimated x_{100} also depends on the assumed d whereas because this parameter is determined primarily by the initial breaks, we would expect it to be independent of d. Variations in the experimental conditions may be responsible for this – specimen C being different from the rest, and the whole set of test pieces being weaker than the others used in the experiment.

Now we turn to the data with 0.5 mm spacing. An initial failure analysis was not appropriate in this case as the majority of breaks are likely to be dependent on others. Inspection of the minimum breaking strains suggested that the characteristics of these test pieces were more in keeping with the general case than were the 1.0 mm samples. Therefore, a first analysis was performed in which the Weibull parameters were assumed the same as for the 1.5 mm data, F alone being estimated. This led to the results given in Table VII. In this case the results seem much more satisfactory in terms of obtaining estimates of F which are consistent across different data sets, and with relatively narrow confidence intervals. In this case, of course, there is much higher dependence in the data, so it is not surprising that we are able to estimate the structure of that dependence more reliably than in the previous two cases.

TABLE VI

d	$\hat{F}$	$\hat{x}_d$	$\hat{x}_{100}$
1.0	80	4.12	3.53
2.0	36	4.08	3.58
2.5	120	3.99	3.53
4.0	46	3.995	3.59
5.0	150	3.88	3.51

As an example of the joint estimation of x_d and F, some values of the negative log likelihood are given in Table VIII. The 95% confidence region consists of all (F, x_d) combinations with negative log likelihood within 1.9 of the minimum, i.e. everything within the range (767.0, 768.9). Combined estimates for various values of d are given in Table IX. In this case a very satisfactory set of results is obtained, with $\hat{F}$ increasing gradually with d (as is to be expected) and the values of $\hat{x}_l$ consistent both with each other and the earlier value 4.46.

6. Discussion

The purpose of this analysis is two-fold. First, it provides an indirect means of determining the stress concentration factors, which are necessary "input" to statistical theories of composite strength as in Batdorf [8] and Smith *et al.* [10]. Secondly, by modelling the whole failure process rather than just the final strength of the bundle, it provides a much firmer basis than previous studies for evaluating the success of the statistical theory for composites. The principal drawbacks of the method, as it has been presented in this paper, are that the variability of estimates of F has been rather large (only in the 0.5 mm case did we obtain fully satisfactory results), and the results are restricted to a rather narrow class of models.

The principal method of computing stress concentrations in the neighbourhood of a reinforcing fibre is shear lag analysis, introduced by Cox [19] and Dow [20]. Hedgepeth [21] calculated stress concentration factors $k(r)$, for the fibre nearest a group of r consecutive failed fibres in a linear array. By his

TABLE VII

Data set	$d = 2$	$d = 2.5$	$d = 4$	$d = 5$
	$x_d = 4.38$	$x_d = 4.35$	$x_d = 4.28$	$x_d = 4.25$
	$\hat{F}$ [conf. int.]	$\hat{F}$ [conf. int.]	$\hat{F}$ [conf. int.]	$\hat{F}$ [conf. int.]
A	10 [10, 11]	11 [10, 11]	13 [12, 14]	11 [10, 11]
B	14 [13, 15]	14 [13, 15]	17 [16, 19]	16 [15, 18]
C	18 [17, 19]	19 [18, 21]	20 [19, 22]	24 [23, 27]
D	14 [13, 15]	15 [14, 16]	17 [16, 19]	17 [16, 19]
E	14 [13, 16]	14 [13, 15]	17 [15, 20]	16 [15, 19]
F	17 [15, 20]	17 [16, 18]	20 [18, 22]	22 [19, 25]

TABLE VIII

x_d	F							
	13	14	15	16	17	18	19	20
4.26								
4.28				804.4	788.8	783.5	784.7	
4.30					775.4	775.3		
4.32				772.0	769.4	773.7	782.4	
4.34			794.2	767.0	769.6	778		
4.36		783.9	768.9	768.1	775.4			
4.38		776.8	769.7	774.5				
4.40		775.9	775.8					

0.50 mm spacing, $d = 2.5$, $w = 30$, all samples combined.

calculations,

$$k(r) = \prod_{j=1}^{r} \frac{2j+2}{2j+1}$$

Note that this does not depend on the ratio of fibre elastic modulus to matrix shear modulus, nor on fibre volume fraction. Hedgepeth's calculations, however, were based on the assumption that the matrix carries only shear stress and no tensile stress at all. In the present context, we are in effect treating the glass–epoxy compound as the matrix and the carbon tows as the fibres, but in this case Hedgepeth's assumption is clearly not satisfied. An alternative analysis taking the matrix elastic strength into account was provided by Fukuda and Kawata [22]. We shall make some comparisons with the Fukuda–Kawata results after first considering another aspect, the effect of the chain-of-bundles model and in particular the choice of d.

It is recognized that the chain of bundles model, in which the material is assumed to be broken up into independent sections of length d, is itself rather a crude approximation which effectively assumes that the stress concentrations around a break form a step function, being $k = 1 + g(r)$ up to a distance $d/2$ away from the break and thereafter 1. In reality the stress concentration factors in the adjacent fibres decay continuously from a maximum opposite the break. A suitable functional form of this relation is that the true stress concentration is $1 + a \exp(-ct)$ at a distance t from the break, i.e. an exponential decay of stress concentration. When the ineffective length d is assumed, this is being approximated by a step function as shown in Fig. 5. It therefore seems reasonable to see how closely the fitted values of F fit such an exponential stress-decay relation.

A suitable criterion for k is obtained by equating the mean numbers of failures in the fibre under the step-function and exponential-decay models. This leads to the equation

$$\delta k^{w} = \int_0^{\delta} [1 + a \exp(-ct)]^{w} \, dt$$

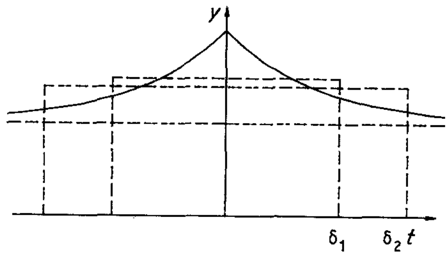

Figure 5 Exponential decay of stress concentration in a fibre adjacent to a broken fibre.

$\delta = d/2$ being the overload region. In the case $r = 1$, the value of F is then obtained from $k = 1 + 1/F$. Recall from Section 5 that, in the case of the 0.5 mm data, we obtained $F = 15$, 16, 19 and 21 corresponding to $d = 2$, 2.5, 4 and 5, respectively. By trial and error we found a very good match with $a = 0.0936$, $c = 0.825$; exact results as shown ($w = 30$) in Table X. This shows that the variation of results with different values of d (or δ) corresponds very closely with what we would expect from an exponential decay model.

A related question concerns the choice of the best value of δ for the chain-of-bundles analysis. Following the early work of Rosen [16] and Zweben and Rosen [23], a value of d in the neighbourhood of 5 to 10 fibre diameters is usually assumed, though the work of Fukuda and Kawata [22] for a matrix bearing tensile stress suggests a value substantially larger. In our case the fibres are replaced by tows of about 0.3 mm diameter, which suggests δ in the range 0.75 to 1.25 mm.

One criterion suggested by the exponential decay model is to base the choice of δ on the positions of new breaks if they occur. The function $[1 + a \exp(-ct)]^{w} - 1$ represents the mean density of new breaks appearing as a result of the stress overload. Hence the proportion of new breaks appearing within δ of the original break is

$$\frac{\int_0^{\delta} \{[1 + a \exp(-ct)]^{w} - 1\} \, dt}{\int_0^{\infty} [\{1 + a \exp(-ct)\}^{w} - 1] \, dt}$$

Evaluating this with $a = 0.0936$, $c = 0.825$ yields proportions 0.75, 0.81, 0.91, 0.95 corresponding to $\delta = 1$, 1.25, 2, 2.5. Thus taking $\delta = 2$, for instance, has the (rough) interpretation that there is over a 90% chance that any new break as a result of the stress overload will occur within δ of the old break.

So far this discussion is restricted to the case when r, the number of consecutive failed elements, is 1. Repeating all this for larger values of r does not yield such good results. For $r = 2$, for instance, the best fit is obtained when $a = 0.136$, $c = 1$ (leading to $F = 15.01$, 16.02, 18.85, 20.62) and corresponding

TABLE IX

d	$\hat{F}$	$\hat{x}_d$	$\hat{x}_1$
2.0	15	4.37	4.47
2.5	16	4.35	4.48
4.0	19	4.26	4.46
5.0	21	4.22	4.45

TABLE X

δ	F
1.0	14.95
1.25	15.98
2.0	19.02
2.5	21.03

proportions (as in the previous paragraph) 0.86, 0.90, 0.96, 0.98 – suggesting that the ineffective length for $r = 2$ is shorter than for $r = 1$, contrary to what has been assumed in previous studies. The corresponding results for $r = 3$ are $a = 0.17$, $c = 1.2$ leading to $F = 15.13, 16.11, 18.75, 20.34$ at our four values of δ and proportions of 0.93, 0.96, 0.986, 0.993. It is somewhat paradoxical that c is increasing with r and this may suggest some discrepancy in our analysis for $r > 1$.

It should also be possible to explain the variation of F with inter-tow distance in relation to existing theories on stress overload, though this is not so easy in view of the fact that we only obtained satisfactory results in the 0.5 mm case and the existing theory may not be directly applicable to the carbon-glass hybrid being considered. As mentioned above, Fukuda and Kawata [22] developed a theory of stress concentrations in composites in which both fibre and matrix bear tensile load. Their results depend on both the ratio of elastic moduli for fibre and matrix, and the fibre volume fraction. In the present study, in which the "matrix" is represented by glass–epoxy composite, the ratio of elastic moduli is very nearly 3. The "volume fraction" is less well defined, because the carbon is surrounded by a considerable quantity of glass–epoxy, the true volume fraction of carbon is near 0, but within the plane where most of the action takes place the carbon tows are juxtaposed with glass–epoxy sections in a "phase-ratio" of about $0.25{:}h$ where h is the separation (centre to centre) between tows. Referring to Fig. 8 in Fukuda and Kawata [22] for the case $r = 1$ and taking the ratio of carbon–glass moduli as 3, it appears that the maximum stress ratio $1 + a$ is about 1.1 for volume fraction 50% and 1.04 for volume fraction 25%, corresponding to inter-tow distances 0.5 and 1.0 mm. Noting that we found $a = 0.0936$ above, this seems consistent. Our results for 1.0 mm spacing are so variable as not to permit the detailed analysis made for the 0.5 mm case, but if we assume $a = 0.4$ and keep the same value of c (0.825) then we obtain $F = 30, 36, 39, 48$ and 55 for the five values of d considered in Section 5. The second and fourth of these values are very close to those quoted in Section 5, but the others are quite different. For the 1.5 mm data, taking $a = 0.013$, $c = 0.825$ yields $F = 94, 122, 154$ for $d = 1, 2.5, 4$, compared with the estimated values 100, 120, 170 in Section 5. Here we have assumed the same value of c (though these results are not very sensitive to c) and the value of a is somewhat lower than the value of about 0.02 suggested by the figure of Fukuda and Kawata – a discrepancy that may be due to the stresses being distributed over a larger volume of the glass–epoxy and hence the effective volume fraction being less than 17%. In subsequent papers, Fukuda and co-workers (see, e.g. Fukunaga *et al.* [23]) have made detailed computations of stress concentration factors in hybrid composites and it may be that their results would cast further light on these computations, but we have not attempted that here.

In conclusion, the statistical analysis presented in this paper has the potential to provide new insight into stress concentration factors and into the applicability of statistical models for composites. However, we obtained fully satisfactory results only in the case of the 0.5 mm data. With the 1.0 mm data, there is evidence of experimental variability that may have made the results non-comparable with the other cases, while with the 1.5 mm data the correlations between the different tows were so slight that there was inevitably huge uncertainty about the estimated value of F. Throughout the paper we have restricted attention to a single statistical model, with F the only parameter describing the stress concentrations. This was dictated partly by computational considerations (difficulty in optimizing over more than one parameter), and partly by the available data, which would have made it difficult to estimate more parameters with a reasonable degree of precision. In principle, however, the method may be applied to a far wider class of models, and this is one aspect which suggests itself for further research.

Acknowledgements
We thank Dr Michael Pitkethly for providing the data, and also Mr Michael Bader and Dr David Clarke for many conversations on this and related topics. L.C.W. was supported by an SERC research grant.

References
1. H. E. DANIELS, *Proc. R. Soc. Lond. A* **183** (1945) 405.
2. D. G. HARLOW and S. L. PHOENIX, *J. Compos. Mater.* **12** (1978) 195.
3. *Idem, ibid.* **12** (1978) 314.
4. D. G. HARLOW and S. L. PHOENIX, *Int. J. Fract.* **17** (1981) 347.
5. *Idem, ibid.* **17** (1981) 601.
6. P. W. MANDERS and M. G. BADER, *J. Mater. Sci.* **16** (1981) 2233.
7. *Idem, ibid.* **16** (1981) 2246.
8. S. B. BATDORF, *J. Reinf. Plast. Compos.* **1** (1982) 153.
9. M. G. BADER and A. M. PRIEST, "Statistical aspects of fibre and bundle strength in hybrid composites", in "Progress in Science and Engineering of Composites", edited by T. Hayashi, K. Kawata and S. Umekawa, Proceedings of ICCM-IV, Tokyo (1982) 1129–36.
10. R. L. SMITH, S. L. PHOENIX, M. GREENFIELD, R. B. HENSTENBURG and R. E. PITT, *Proc. R. Soc. A* **388** (1983) 353.
11. M. G. BADER and M. J. PITKETHLY, "Probabilistic aspects of the strength and modes of failure of hybrid fibre composites", in "Mechanical Characterisation of Fibre Composite Materials", edited by R. Pyrz (Aalborg University, Denmark, 1986).
12. M. G. BADER, M. J. PITKETHLY and R. L. SMITH, "Probabilistic models for hybrid composites", Vol. 5, Proceedings of ICCM-VI and ECCM-2, edited by F. L. Matthews, N. C. R. Buskell, J. M. Hodgkinson and J. Morton (Elsevier Applied Science, London, 1987).
13. D. R. COX and P. A. W. LEWIS, "The Statistical Analysis of Series of Events" (Chapman and Hall, London, 1966).
14. W. WEIBULL, *Proc. Ing. Vetenskapsakad* **151** (1939).
15. A. S. WATSON and R. L. SMITH, *J. Mater. Sci.* **20** (1985) 3260.
16. B. W. ROSEN, *AIAA J.* **2** (1964) 1985.
17. C. ZWEBEN, *AIAA J.* **6** (1968) 2325.
18. M. J. PITKETHLY, PhD thesis, University of Surrey, Guildford, unpublished.
19. H. F. COX, *Brit. J. Appl. Phys.* **3** (1952) 72.
20. N. F. DOW, "Study of stresses near a discontinuity in a

filament reinforced composite model", General Electricity Space Sciences Laboratory Report No. R63 SD 61 (1963).
21. J. M. HEDGEPETH, "Stress concentrations in filamentary structures", Technical Note D-882, NASA, Washington (1961).
22. H. FUKUDA and K. KAWATA, *Fibre Sci. Technol.* **9** (1976) 189.
23. C. ZWEBEN and B. ROSEN, *J. Mech. Phys. Solids* **18** (1970) 189.
24. H. FUKUNAGA, T.-S. CHOU and H. FUKUDA, *J. Reinf. Plast. Compos.* **3** (1984) 145.

Received 14 March
and accepted 27 July 1988